

This work is dedicated to the memory of Asher Peres, teacher and friend, whom we shall always greatly miss.

Quantum estimation of relative information

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We derive optimal schemes for preparation and estimation of relational degrees of freedom between two quantum systems. We specifically analyze the case of rotation parameters representing relative angles between elements of the $SU(2)$ symmetry group. Our estimation procedure does not assume prior knowledge of the absolute spatial orientation of the systems and as such does not require information on the underlying classical reference frame in which the states are prepared.

I. INTRODUCTION

Estimating the state of a given quantum system is a fundamental primitive of many quantum information tasks. This problem is usually translated to the estimation of the value of a physical parameter describing specific properties of the preparation procedure. In many instances [1, 2, 3, 4, 5, 6, 7, 8, 9] global parameters of the state space define a natural scheme for encoding quantum information. The global parameters describe collective degrees of freedom of a system with respect to the external environment and are often related to an overall symmetry transformation of the state.

However, encoding information into global degrees of freedom may be often problematic,

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due to lack of knowledge of the reference frame with respect to which they were prepared, or due to collective decoherence by which they are affected. Encoding information into relative degrees of freedom, possible whenever a quantum system is decomposable into parts, can overcome many of the difficulties encountered in these situations. Such an encoding scheme has been demonstrated experimentally [10, 11] and can be applied to quantum computation [12], communication [13, 14] and cryptography [15, 16].

The aim of this work is to develop efficient preparations and measurement schemes for the relative parameters describing symmetries between different components of a system. We note that such measurements can induce relative relations when previously absent, as in the case of the relative phase between two Fock states or the relative position between two momentum eigenstates [17].

In this paper we specifically confront the task of efficient estimation of relative rotation angles between two representation vectors of the $SU(2)$ symmetry group. This problem was first addressed by Bartlett *et. al.* [18], who explicitly worked out the estimation of $SU(2)$ rotation angles between two spin coherent states. In this article we proceed along the lines of earlier work that Asher Peres started together with us, and propose an extension of the previous methods. Our approach is based on an optimization of the quantum states used in such protocols with respect to some average measure of success of the estimation task, which we shall refer to as the fidelity. The key to this problem lies in the decomposition of both the signal and the measurement elements in irreducible components, invariant under global rotation transformations.

In the following section we discuss the general mathematical structure of the problem. In section III A we derive the optimal measurement for the case in which one system is comprised of two spin-1/2, and the other of one spin-1/2. We find that preparing two spin-1/2 parallel to each other leads to a marginally higher fidelity than the antiparallel case; then we determine an optimal preparation procedure which gives a higher fidelity than the ones achieved by the above preparations. This is in contrast to the known results for transmitting a spatial direction [2]. We then proceed in section III B by replacing the single spin state of the second system with a spin- j coherent state and determine the optimal preparations for the cases of anti-parallel and parallel spins, which now yield nearly the same fidelity for any value of j . We then study the quantum/classical correspondence by considering the limit in which j becomes very large. In section III C it is shown that in this limit the

problem reduces to estimating the first state with respect to a classical reference direction. Our results establish the correspondence between relative degrees of freedom in quantized systems and collective degrees of freedom defined with respect to a classical reference frame.

II. ESTIMATION OF RELATIVE PARAMETERS

A. Formulation of the general problem and basic notations

We begin this section by formulating the problem for a general symmetry group and introducing the basic notation that is useful for our general scheme. In the following, let G denote a symmetry group, compact or finite, that describes the global properties of the system through its action on a set of parameters T . We shall consider $G = SU(2)$ acting on quantum spin states, parametrized by the set of rotation angles. The possible states of the system are pure states $\Psi(\Theta)$, $\Theta \in T$ in a d -dimensional Hilbert space \mathcal{H} carrying a unitary representation $\{U(g), g \in G\}$ of G . To introduce relative symmetry transformations, we assume that the representation space \mathcal{H} is a tensor product of two components $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ of dimensions d_1, d_2 respectively. The representation of G on \mathcal{H} is decomposed into the product $\{U_1(g) \otimes U_2(g)\}$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$. This product representation of $G \times G$, in which each component is transformed by the same element $g \in G$, is isomorphic to G itself. We introduce another set, t , of parameters θ , to describe a relative symmetry between the two components, represented by a group of transformations \tilde{G} , which can be the same G as before or a subgroup of the latter. \tilde{G} refers to a symmetry property of one of the subsystems (say 2) with respect to the other (say 1), such as, in our case, a relative rotation angle. We call $U_2(h)$ its representation operators on the space \mathcal{H}_2 . We shall restrict ourselves to the case where each of the subsystems is prepared in a pure state, although the formalism can easily be extended to mixed states. The total state on \mathcal{H} can be written in terms of the two sets of parameters as $\Psi(\Theta, \theta)$. Its transformation under a global operation is

$$U(g)\Psi(\Theta, \theta) \equiv U_1(g) \otimes U_2(g)\Psi(\Theta, \theta) = \Psi(g\Theta, \theta). \quad (1)$$

The objective of the construction is to define an efficient estimation procedure for the relative parameters θ , overlooking the information carried by the global parameters Θ . Note that we might be interested only in estimating a subset of the relative parameters (which will be the case in the following sections). In order to quantify the efficiency of our estimation

procedure, we choose a utility function $f(\mu, \theta)$ which measures the deviation of the estimated parameters μ from their true values θ . We consider only utility functions which are invariant under global rotations. The measurement apparatus, represented by the POVM $\{E_\mu\}$, should be constructed such that it maximizes the average fidelity, denoted by F , and given by

$$F\{E_\mu\} = \sum_\mu \int d\theta d\Theta P(\Theta, \theta) \text{Tr}[\rho(\Theta, \theta) E_\mu] f(\mu, \theta), \quad (2)$$

where $P(\Theta, \theta)$ is a prior probability distribution over the global and relative parameters, and $\rho(\Theta, \theta) = |\Psi(\Theta, \theta)\rangle\langle\Psi(\Theta, \theta)|$. As noted in [18], we can assume that the global and relative parameters are independent random variables, and that the global parameter is uniformly distributed on its domain of definition, *i.e.*,

$$P(\Theta, \theta) d\theta d\Theta = p(\theta) d\theta d\Theta. \quad (3)$$

Now, using the definition for the global transformation (1) and the properties of the trace, we can write

$$F\{E_\mu\} = \sum_\mu \int d\theta p(\theta) \text{Tr}[\bar{\rho}(\theta) E_\mu] f(\mu, \theta), \quad (4)$$

where

$$\bar{\rho}(\theta) = \int dg U^\dagger(g) \rho(\Theta, \theta) U(g). \quad (5)$$

In the last equation, dg is the invariant measure for the group G . As a consequence of equations (4) and (5), we need only to consider a reduced form of the input state which is manifestly invariant under global transformations. Schur's lemma [20] then assures that the input state is block diagonal in the irreducible representations of G ,

$$\bar{\rho}(\theta) = \sum_{\oplus J} \rho^{(J)}(\theta). \quad (6)$$

The above considerations also have implications on the form of the optimal measurement. In fact, using the global invariance of $\bar{\rho}$, we have

$$\begin{aligned} F\{E_\mu\} &= F\{U(g) E_\mu U^\dagger(g)\} \\ &= F\left\{\int dg U(g) E_\mu U^\dagger(g)\right\}. \end{aligned}$$

This relation implies that when searching for the optimal estimation procedure it suffices to consider POVMs which are invariant under a global transformation

$$U(g) E_\mu U^\dagger(g) = E_\mu, \forall g \in G. \quad (7)$$

Equation (7) is by itself a strong prerequisite on the structure of the POVM elements. Indeed, if combined with Schur's lemma, it implies that whenever the total Hilbert space \mathcal{H} can be decomposed into a direct sum of irreducible representations under the global transformation, the optimal POVM elements labelling the different outcomes have the simple form

$$E_\mu = \sum_J p_{\mu,J} E_\mu^J, \quad (8)$$

where each of the operators E_μ^J has support only on the representation labelled by J and $\sum_\mu p_{\mu,J} E_\mu^J = \mathbb{1}_J$. Here the symbol $\mathbb{1}_J$ denotes the identity operator in the J subspace. The search for optimal POVMs may be further restricted to the case in which all elements have support on only one representation space, due to the linearity of the fidelity functional (given a POVM of the form (8), the POVM with elements $E_{\mu,J} \equiv p_{\mu,J} E_\mu^J$ yields the same fidelity as the original one by linearity of the trace and obeys the restriction that each element has support on only one representation).

In the following sections, the general considerations outlined above will be applied to the problem of transmitting relative rotation angles of the $SU(2)$ group.

B. Estimation of a relative angle between spin coherent states

In estimating relative, as opposed to absolute, rotation angles, we assume no prior knowledge of the overall orientation of the classical frame in which the system is defined. Following the notation introduced above, the problem may be illustrated as follows.

Imagine that, with no prior knowledge on the absolute spatial orientation of two observers, Alice and Bob, we were requested to estimate the angle β between two unit vectors $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$, each chosen by one of them, by measuring a pair of $SU(2)$ spin states prepared in the corresponding reference frame of each observer. This task is in general possible owing to the fact that states belonging to an $SU(2)$ representation space can be used as intrinsic direction indicators [1, 2, 3, 4, 5, 6, 7, 8, 9] and therefore it makes sense to consider relative angles between them.

The simplest way to achieve the task would be to consider two $SU(2)$ coherent states, corresponding, say, to spins j_1, j_2 ,

$$\hat{\mathbf{n}}_1 \cdot \mathbf{J}_1 |\psi_1\rangle = j_1 |\psi_1\rangle, \quad \hat{\mathbf{n}}_2 \cdot \mathbf{J}_2 |\psi_2\rangle = j_2 |\psi_2\rangle. \quad (9)$$

Without loss of generality, we can assume that Alice and Bob choose the z -axis of their reference frame. A state denoted by $|\psi_2\rangle$ in Bob's frame is written in Alice's reference frame as $U_2^z(\alpha)U_2^y(\beta)U_2^z(\gamma)|\psi_2\rangle$, where $U_k^x(\alpha) = e^{i\mathbf{J}_k \cdot \mathbf{x}\alpha}$, and similarly for the other directions. The angles α, β, γ are the three Euler angles relating Alice's reference frame to the one of Bob. Note that the angle $\beta \in [0, \pi]$ is also the angle between $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$. We introduce a global reference frame which is rotated with respect to Alice's frame by an angle α around the z -axis. In this frame the composite state is given by

$$|\Psi(\alpha, \beta, \gamma)\rangle = U_1^z(\alpha)|\psi_1\rangle \otimes U_2^z(\beta)U_2^z(\gamma)|\psi_2\rangle. \quad (10)$$

For spin coherent states, as in Eq. (9), the above equation is simplified to

$$|\Psi(\beta)\rangle = |j_1, m_1 = j_1\rangle \otimes U_2^y(\beta)|j_2, m_2 = j_2\rangle, \quad (11)$$

up to an overall phase. Notice that in Eq. (10),(11) we have implicitly specified the global parameter Θ , which we shall omit from now on from our notation. So far we have a set of three relative parameters, $\theta = \{\alpha, \beta, \gamma\}$, with a joint probability distribution given by the Haar-measure

$$p(\alpha, \beta, \gamma) = \frac{1}{8\pi^2} \sin \beta, \quad (12)$$

which corresponds to a random orientation of Alice's and Bob's reference frames. Since the party making the measurement (Bob) is not interested in estimating $\alpha, \gamma \in [0, 2\pi]$, these parameters are averaged out by integrating over their range. We are then left with the probability distribution $p(\beta) = \sin \beta/2$, which corresponds to the probability density of the angle between two random unit vectors in three dimensions.

Let us denote by $\mathcal{H}_1, \mathcal{H}_2$ the Hilbert spaces of the systems prepared by Alice and Bob respectively, carrying the $SU(2)$ representations j_1 and j_2 . The composite Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ carries a diagonal product representation of $SU(2) \otimes SU(2)$, $U^{j_1}(g) \otimes U^{j_2}(g)$, $g \in SU(2)$, which corresponds to a *global* symmetry operation labelled by the parameter g , acting identically on the two subspaces. This representation may be reduced as $\sum_{\oplus J=|j_1-j_2|}^{j_1+j_2} \mathcal{H}_J$, where each component has multiplicity one. Invariance of the measurement operators under a global rotation (as explained in the previous section) reduces the signal state to the form

$$\bar{\rho}(\beta) = \sum_J p_J(\beta) \Pi_J, \quad (13)$$

where Π_J are projectors on the representations J . Since each representation of the global rotation (specified above by J) appears *only once* in the signal state, the measurement process amounts to estimating a probability distribution over the relative angle β . The scenario described above, where each of the parties prepares a spin coherent state, is the one examined in [18].

III. GENERAL SCHEME

In the procedure outlined in the last section, the two parties, Alice and Bob, use spin coherent states in order to indicate their chosen direction. However, for the task considered here this is not the optimal preparation. It is known, in fact, that optimal $SU(2)$ direction indicators exploit entanglement between components belonging to different irreducible representations [3, 4, 5, 6, 7]. This suggests to consider the following general encoding procedure. Let

$$|\Phi\rangle = \sum_{j_1=0}^{j_{max}} \sum_{m_1=-j_1}^{j_1} a_{m_1}^{j_1} |j_1 m_1\rangle \quad (14)$$

be a generic state in \mathcal{H}_1 . By choosing a unit vector $\hat{\mathbf{n}}_1$, Alice would prepare the state $U(\hat{\mathbf{n}}_1)|\psi_1\rangle$,

$$|\psi_1\rangle = U(\hat{\mathbf{n}}_1)|\Phi\rangle, \quad (15)$$

where $U(\hat{\mathbf{n}}_1)$ is a unitary operator corresponding to the rotation which carries Alice's $\hat{\mathbf{z}}$ -axis onto $\hat{\mathbf{n}}_1$. Our goal is to find the optimal state $|\Phi\rangle$ for the case in which Bob indicates his direction $\hat{\mathbf{n}}_2$ with a coherent state $|\psi_2\rangle$ satisfying

$$\hat{\mathbf{n}}_2 \cdot \mathbf{J}_2 |\psi_2\rangle = j_2 |\psi_2\rangle. \quad (16)$$

Note that equations (15) and (16) are written in Alice's and Bob's reference frames, respectively. In a global reference frame, specified as in the last section, the total state is given by

$$\begin{aligned} |\Psi(\alpha, \beta, \gamma)\rangle &= U_1^z(\alpha) |\psi_1\rangle \otimes U_2^y(\beta) U_2^z(\gamma) |\psi_2\rangle \\ &= \sum_{j_1 m_1} a_{m_1}^{j_1} e^{i m_1 \alpha} |j_1 m_1\rangle \otimes U_2^y(\beta) e^{i j_2 \gamma} |j_2 j_2\rangle. \end{aligned} \quad (17)$$

As before, the state $|\Psi\rangle$ is expressed in terms of an arbitrary orientation of the global reference frame, so that the parameter Θ may be omitted. However, one can see that the

angles α, γ now induce relative phases between the different components of the state. Since our protocol does not deal with the estimation of α and γ (only the angle β is considered), we will average over them in the expression of the fidelity

$$F = \sum_{\mu} \int d_{\alpha\beta\gamma} dg \text{Tr} \left[U(g) \rho(\alpha, \beta, \gamma) U^{\dagger}(g) E_{\mu} \right] f(\mu, \beta), \quad (18)$$

where we denoted $d_{\alpha\beta\gamma} \equiv 1/8\pi^2 \sin(\beta) d\alpha d\beta d\gamma$. Integrating over α and γ , we have

$$\frac{1}{4\pi^2} \int d\alpha d\gamma \rho(\alpha, \beta, \gamma) = \sum_{m_1} c_{m_1} \rho_{m_1} \otimes U_2^y(\beta) |j_2 j_2\rangle \langle j_2 j_2| U_2^{y\dagger}(\beta), \quad (19)$$

with

$$\rho_{m_1} = |\psi_{m_1}\rangle \langle \psi_{m_1}|, \quad |\psi_{m_1}\rangle = \sum_{j_1=m_1}^{j_{max}} a_{m_1}^{j_1} |j_1, m_1\rangle, \quad (20)$$

and c_{m_1} given by

$$c_{m_1} = \sum_{j_1} |a_{m_1}^{j_1}|^2, \quad \sum_{m_1} c_{m_1} = 1. \quad (21)$$

In the following we will search for the optimal generic state $|\Phi\rangle$. Note that after integrating over α, γ we get a convex combination of states with different m_1 . Thus the fidelity will contain a linear combination of contributions from the different m_1 sectors,

$$F = \sum_{m_1} c_{m_1} F(m_1) \leq \max_{m_1} F(m_1). \quad (22)$$

From the above discussion it is clear [19] that the optimal generic state can be taken with m_1 fixed, *i.e.*, of the form

$$|\Phi_{m_1}\rangle = \sum_{j_1=m_1}^{j_{max}} a_{m_1}^{j_1} |j_1 m_1\rangle. \quad (23)$$

In the following we shall restrict ourselves to generic states of this form. The rotation in Eq. (17) by the relative angle $\beta \in [0, \pi]$ is expressed in the standard Euler angle notation as a rotation around the y -axis, given by the matrix

$$U_2^y(\beta) |j_2 j_2\rangle \equiv \sum_m d_{m j_2}^{j_2}(\beta) |j_2 m\rangle, \quad (24)$$

where the $d_{m' m}^{j_2}(\beta)$ can be expressed using Jacobi polynomials (see for example [21]). The superscript in the above equation refers to the $(2j_2 + 1)$ -dimensional irreducible representation of spin j_2 . The signal state (17) is given explicitly (up to an overall phase) by

$$|\Psi(\beta)\rangle = \sum_{j_1} a_{m_1}^{j_1} \sum_{m_2} d_{m_2 j_2}^j(\beta) |j_1 m_1, j_2 m_2\rangle, \quad (25)$$

where $|j_1 m_1, j_2 m_2\rangle \equiv |j_1 m_1\rangle \otimes |j_2 m_2\rangle$. Note that unlike the example discussed in section II B and in [18], we now exploit repeated irreducible representations J , since each value of j_1 gives rise to a series of total angular momentum $J = |j_1 - j_2|, \dots, (j_1 + j_2)$. The equivalent repetitions of the representation J are labelled by j_1 , with j_2 being fixed. The state can be written in the basis $(J M, j_1 j_2)$ using the unitary transformation

$$|j_1 m_1, j_2 m_2\rangle = \sum_{J M} C_{j_1 m_1 j_2 m_2}^{J M} |J M, j_1 j_2\rangle, \quad (26)$$

with C denoting the Clebsch–Gordan coefficients, and M denoting the z component of the total angular momentum.

We now need to compute the averaged state $\bar{\rho}(\beta)$, following Eq. (5). A corollary to Schur’s lemma [20] then states that, for irreducible representations σ, τ of a group G , a group-averaged operator satisfies

$$\int dg U^{(\sigma)}(g) A U^{(\tau)}(g)^\dagger = \frac{\delta_{\sigma, \tau} \text{Tr}_\sigma(A) \mathbb{1}}{d_\sigma}, \quad (27)$$

where $\delta_{\sigma, \tau}$ is a Kronecker delta over the inequivalent representations and the trace is computed over the d_σ -dimensional space of the irreducible representation σ . The above corollary is applied to obtain the invariant reduced density operator $\bar{\rho}(\beta)$. Remembering that the global rotation operator $U(\Omega)$ is a direct sum of operators $U(\Omega) = \oplus_J U^{(J)}(\Omega)$, we see from Eq. (27) that $\bar{\rho}(\beta)$ is also block diagonal in the representations J ,

$$\begin{aligned} \bar{\rho}(\beta) &= \int U(\Omega) \rho(\beta) U^\dagger(\Omega) d\Omega \\ &= \sum_{\oplus J} \bar{\rho}^{(J)}(\beta), \end{aligned} \quad (28)$$

where the operators $\rho^{(J)}(\beta)$ bear the indices j_1, j'_1 and are given by

$$\bar{\rho}^{(J)}(\beta)_{j_1, j'_1} = \sum_M \langle J, M, j'_1 | \rho(\beta) | J, M, j_1 \rangle. \quad (29)$$

Note that by averaging over the global rotation one does not diagonalize the operator $\bar{\rho}(\beta)$ with respect to the additional quantum number j_1 . The invariant signal state $\bar{\rho}(\beta)$ is block diagonal over the irreducible representations with off diagonal elements across the repeated ones

$$\bar{\rho}(\beta) = \sum_{\oplus J} \sum_{j_1 j'_1} \frac{1}{(2J+1)} \sum_{M m_2} a_{m_1}^{j_1} a_{m_1}^{*j'_1} \left(d_{j_2 m_2}^{j_2}(\beta) \right)^2 C_{j_1 m_1 j_2 m_2}^{J M} C_{j'_1 m_1 j_2 m_2}^{J M} |J, j_1\rangle \langle J, j'_1|. \quad (30)$$

Remember that m_1 is fixed.

Note that the state $\bar{\rho}(\beta)$ does not, even implicitly, depend on the orientation of the reference frame of the measurement apparatus. We will now use $\bar{\rho}(\beta)$ to determine the state $|\psi_1\rangle$ which will enable optimal estimation of β . To this end, we fix a convenient figure of merit as measure of the discrepancy between the estimated and the given value of $\beta \in [0, \pi]$, namely the quadratic utility function $f(\mu, \beta) = \cos^2((\mu - \beta)/2)$, where μ is the estimated value of the parameter. The choice of the utility function is not unique, and a different choice might lead to different optimal states and POVMs. However, the optimization procedure, as described below, is independent of this choice. The above utility function has the advantage of having been broadly used throughout earlier literature [1, 2, 3, 4, 5, 6, 7, 8].

We denote the POVM elements by $\{E_\mu\}$, with $\mu \in [0, \pi]$, and $\sum_\mu E_\mu = 1$. The average fidelity with respect to the given figure of merit, integrating over all possible transmitted angles β and all possible inferred values μ , is

$$F[\{\mu\}, \{E_\mu\}] = \sum_\mu \int \text{Tr}[\bar{\rho}(\beta) E_\mu] \cos^2\left(\frac{\mu - \beta}{2}\right) \sin \beta d\beta/2. \quad (31)$$

Note that the fidelity is a functional both of the set of estimates $\{\mu\}$ and of the POVM used for the estimation procedure $\{E_\mu\}$, where to each estimate corresponds a

(single) POVM element. The probability of estimating μ for a true angle β is $\text{Tr}[\bar{\rho}(\beta) E_\mu]$.

The above expression can be rewritten by exchanging the order of the integral with the trace (due to the linearity of the integration and finiteness of the sum) and gives the fidelity in the form

$$F[\{\mu\}, \{E_\mu\}] = \sum_\mu \text{Tr}\{A_\mu E_\mu\}. \quad (32)$$

with

$$A_\mu \equiv \int \bar{\rho}(\beta) \cos^2((\mu - \beta)/2) \sin \beta d\beta/2. \quad (33)$$

Since $\bar{\rho}(\beta)$ is block diagonal, also A_μ can be written as a direct sum $A_\mu = \sum_{\oplus J} A_\mu^J$.

In terms of the basis representation states of angular momenta $(J; j_1, j_2)$, and using Eq. (30), the operator A_μ can be explicitly written as

$$A_\mu = \sum_{\oplus J} \sum_{j_1 j_1'} \frac{1}{2J+1} \sum_{M m_2} \left\{ a_{m_1}^{j_1} a_{m_1'}^{*j_1'} I_{m_2}^{j_2}(\mu) C_{j_1 m_1 j_2 m_2}^{J M} C_{j_1' m_1' j_2 m_2}^{J M} \right\} |J, j_1\rangle \langle J, j_1'|, \quad (34)$$

where

$$I_{m_2}^{j_2}(\mu) \equiv \int \left(d_{j_2 m_2}^{j_2}(\beta) \right)^2 \cos^2((\mu - \beta)/2) \sin \beta d\beta/2. \quad (35)$$

This expression may be evaluated using the properties of the Wigner functions $d_{mm'}^j$ and their representation in terms of Jacobi polynomials [21]. The average fidelity can now be written as a sum of contributions from each subspace of given J

$$F[\{\mu\}, \{E_\mu\}] = \sum_J \sum_\mu \text{Tr}(A_\mu^J E_\mu^J), \quad (36)$$

where $\sum_\mu E_\mu^J = \mathbb{1}_J$. Given a set of estimates $\{\mu\}$, the task of maximizing the expression $\sum_\mu \text{Tr}(A_\mu^J E_\mu^J)$ is straightforward, at least numerically (for example, by using semidefinite programming [22]). However, in our approach, in order to maximize the average fidelity (36), we need to maximize over the set $\{\mu\}$, so that the maximal fidelity F_{\max} will actually be given by

$$F_{\max} = \max_{\{\mu\}} \max_{\{E_\mu\}} F[\{\mu\}, \{E_\mu\}]. \quad (37)$$

A. The case of $j_2 = 1/2$, $j_1 \in \{0, 1\}$

We will now solve the optimization problem of Eq. (37) for the following case: the state $|\psi_2\rangle$ is a spin 1/2 coherent state, while $|\psi_1\rangle$ is composed of two spin 1/2 systems, so that $j_1 \in \{0, 1\}$. According to the discussion in the previous section, we can restrict ourselves to two classes of generic states $|\Phi\rangle$

$$|\Phi_0\rangle = a|j_1 = 0, m_1 = 0\rangle + \sqrt{1-a^2}|j_1 = 1, m_1 = 0\rangle, \quad (38)$$

and

$$|\Phi_1\rangle = |j_1 = 1, m_1 = 1\rangle \quad (39)$$

Let us first discuss the case where $|\Phi\rangle = |\Phi_1\rangle$. In this simple case, the state $|\psi_1\rangle$ is just a spin-1 coherent state or, viewed as composed of two spins, it is a polarized state with parallel spins along the vector \hat{n}_1 . Coupling the representations of ψ_1 and ψ_2 gives

$$\mathbf{1} \otimes \mathbf{1}/2 = \mathbf{1}/2 \oplus \mathbf{3}/2,$$

and the operators A_μ^J are one-dimensional, making the optimization trivial. In this case, the optimal measurement simply consists in the projections onto the $J = 1/2$ and $J = 3/2$ subspaces, as there are no repeated representations. The fidelity achieved with this state is $F_{\max}[\text{parallel}] = 0.90983$.

Next, we consider $|\Phi\rangle = |\Phi_0\rangle$. Coupling the representations of ψ_1 and ψ_2 gives in this case

$$(\mathbf{1} \oplus \mathbf{0}) \otimes \mathbf{1/2} = \mathbf{1/2} \oplus \mathbf{1/2} \oplus \mathbf{3/2},$$

and therefore the density matrix $\bar{\rho}(\beta)$ contains two blocks of dimensions 2 and 1 corresponding to $J = 1/2$ and $J = 3/2$, respectively. The operators A_μ^J are given by

$$A_\mu^{1/2} = \begin{pmatrix} \frac{a^2 (4 + \pi \sin \mu)}{8} & \frac{a\sqrt{1-a^2} \cos \mu}{6\sqrt{3}} \\ \frac{a\sqrt{1-a^2} \cos \mu}{6\sqrt{3}} & \frac{6(2-a^2) + 3\pi(1-a^2) \sin \mu}{72} \end{pmatrix}$$

and

$$A_\mu^{3/2} = \frac{12(1-a^2) + 3(1-a^2) \sin \mu}{36}. \quad (40)$$

We are seeking the set μ and E_μ^J which maximizes the mean fidelity (36). Let us start with the $J = 3/2$ subspace. Since this subspace is one-dimensional, the restriction to operators which are invariant under a global rotation leaves us with one operator only, $E_{\mu_{3/2}}^{3/2}$, which is the projection operator on the $J = 3/2$ subspace. The estimate $\mu_{3/2}$ which maximizes the corresponding expression for $A_\mu^{3/2}$ in reference to Eq. (40), is obviously given by $\mu_{3/2} = \pi/2$.

Next, we consider the 2-dimensional subspace of $J = 1/2$. Following [23], we define an operator Υ as

$$\Upsilon = \sum_{\mu} A_\mu E_\mu. \quad (41)$$

For a set $\{\mu\}$, a POVM $\{E_\mu\}$ is optimal if and only if it satisfies the following set of conditions

$$\Upsilon - A_\mu \geq 0 \quad (42)$$

for each μ in the set of estimates $\{\mu\}$, with the additional requirement that Υ be hermitian. The inequality sign in Eq. (42) means that the operator $\Upsilon - A_\mu$ must be positive semi-definite. The maximal fidelity will then be given by

$$F_{\max} = \text{Tr } \Upsilon. \quad (43)$$

In order to see that equations (42) and (41) indeed lead to the maximization of the mean fidelity, consider a different POVM $\{E'_\mu\}$, such that

$$\sum_{\mu} E'_\mu = \mathbb{1}. \quad (44)$$

The difference between the fidelity achieved with this POVM and the one achieved with the optimal one is

$$F_{\max} - F' = \text{Tr} \sum_{\mu} (\Upsilon - A_{\mu}) E'_{\mu}, \quad (45)$$

thanks to Eqs. (43) and (44). Now, if C and D are positive semi-definite hermitian operators, then they satisfy

$$\text{Tr}(CD) \geq 0. \quad (46)$$

Setting $C = \Upsilon - A_{\mu}$ and $D = E'_{\mu}$, we obtain

$$F_{\max} - F' \geq 0, \quad (47)$$

as desired.

Let us first maximize the average fidelity for only two estimates μ_1 and μ_2 , in correspondence to which we have the POVM elements $E_{\mu_1}^{1/2} + E_{\mu_2}^{1/2} = \mathbb{1}_{J=1/2}$. Then

$$\begin{aligned} \Upsilon - A_{\mu_1}^{1/2} &= A_{\mu_1}^{1/2} E_{\mu_1}^{1/2} + A_{\mu_2}^{1/2} E_{\mu_2}^{1/2} - A_{\mu_1}^{1/2} \\ &= (A_{\mu_2}^{1/2} - A_{\mu_1}^{1/2}) E_{\mu_2}^{1/2} \geq 0, \end{aligned} \quad (48)$$

since $\Upsilon - A_{\mu_1}^{1/2}$ is non-negative if the POVM is optimal. Let us denote by η_i and $|\eta_i\rangle$ the eigenvalues and corresponding eigenvectors of the operator $\Delta \equiv A_{\mu_2}^{1/2} - A_{\mu_1}^{1/2}$. For each $|\eta_i\rangle$ we can write

$$\langle \eta_i | (A_{\mu_2}^{1/2} - A_{\mu_1}^{1/2}) E_{\mu_2}^{1/2} | \eta_i \rangle = \eta_i \langle \eta_i | E_{\mu_2}^{1/2} | \eta_i \rangle \geq 0, \quad (49)$$

using (48). If we assume that η_i is negative, then (49) gives

$$\langle \eta_i | E_{\mu_2}^{1/2} | \eta_i \rangle \leq 0;$$

on the other hand, since $E_{\mu_2}^{1/2}$ is positive semi-definite, we must have

$$\langle \eta_i | E_{\mu_2}^{1/2} | \eta_i \rangle = 0, \quad \text{if } \eta_i < 0,$$

and similarly

$$\langle \eta_i | E_{\mu_1}^{1/2} | \eta_i \rangle = 0, \quad \text{if } \eta_i > 0.$$

Thus $E_{\mu_2}^{1/2}$ projects onto the subspace spanned by $|\eta_i\rangle$ with $\eta_i \leq 0$ and $E_{\mu_1}^{1/2}$ projects onto the subspace of positive eigenvalues of the operator Δ . The subspace with $\eta_i = 0$ does not contribute to the fidelity, so that the maximal fidelity is given by

$$F_{\max} = \text{Tr} \Upsilon = \text{Tr} A_{\mu_1}^{1/2} + \sum_{\eta_i \geq 0} \eta_i. \quad (50)$$

Let us now assume that $\mu_1 = \mu$ and $\mu_2 = \pi - \mu$. We shall see that this choice will lead to the optimal measurement for the class of states under consideration. Indeed, we have

$$\Delta = \begin{pmatrix} 0 & \frac{a\sqrt{1-a^2}\cos\mu}{3\sqrt{3}} \\ \frac{a\sqrt{1-a^2}\cos\mu}{3\sqrt{3}} & 0 \end{pmatrix}, \quad (51)$$

with eigenvalues $\pm \frac{a\sqrt{1-a^2}\cos\mu}{3\sqrt{3}}$ and corresponding eigenvectors $|+\rangle = (1, 1)^T$ and $|-\rangle = (1, -1)^T$. The contribution to the fidelity from the $J = 1/2$ subspace $F^{1/2}$ is now given by

$$F^{1/2}(\mu) = \frac{a\sqrt{1-a^2}\cos\mu}{3\sqrt{3}} + \text{Tr } A_\mu^{1/2}. \quad (52)$$

At this point, in order to find the maximal mean fidelity under our assumptions, it suffices to maximize the function $F^{1/2}(\mu)$. A simple calculation shows that the maximum is attained for

$$\nu = \tan^{-1} \left[3\sqrt{3}(1 + 2a^2)\pi / (8a\sqrt{1-a^2}) \right]. \quad (53)$$

It remains to check that indeed the choice

$$\mu_1 = \nu, \quad \mu_2 = \pi - \nu \quad (54)$$

leads to the maximal fidelity for the $J = 1/2$ subspace. A proof of this fact is provided by the following argument. Consider a general set of estimates $\{\boldsymbol{\mu}\} = \{\mu_1, \mu_2, \mu_3, \dots, \mu_n\}$, with a corresponding set of POVM elements $E_\mu^{1/2}$, and let

$$F^{1/2}[\{\boldsymbol{\mu}\}] = \max_{\{E_\mu^{1/2}\}} F^{1/2}[\{\boldsymbol{\mu}\}, \{E_\mu^{1/2}\}] \quad (55)$$

be the maximal fidelity achieved by this set. We would like to show that by adding ν and $\pi - \nu$ to the set $\{\boldsymbol{\mu}\}$ the mean fidelity can never decrease with respect to the optimal bound and, at the same time, the bound is attained by these two values alone. Let $\{\tilde{\boldsymbol{\mu}}\}$ denote the new set obtained by adding ν and $\pi - \nu$, defined by Eq. (53), to the set $\{\boldsymbol{\mu}\}$. The first property,

$$F^{1/2}[\{\tilde{\boldsymbol{\mu}}\}] \geq F^{1/2}[\{\boldsymbol{\mu}\}], \quad (56)$$

simply follows from the fact that adding estimates to a given set can only increase the mean fidelity. To complete the argument we still have to show that

$$F^{1/2}[\{\tilde{\boldsymbol{\mu}}\}] = F^{1/2}[\{\nu, \pi - \nu\}]. \quad (57)$$

The optimal measurement for $\{\nu, \pi - \nu\}$, following the earlier discussion, is defined by the projectors on the negative and positive eigenvectors of the operator $A_\nu^{1/2} - A_{\pi-\nu}^{1/2}$, *i.e.*,

$$E_\nu^{1/2} = |-\rangle\langle-|, \quad E_{\pi-\nu}^{1/2} = |+\rangle\langle+|. \quad (58)$$

Consider a POVM for the set $\{\tilde{\mu}\}$ which consists of the two operators in (58), and of

$$E_\mu^{1/2} = 0 \quad \text{if} \quad \mu \neq \nu, \pi - \nu.$$

This POVM is optimal also for the set $\{\tilde{\mu}\}$. To see this, we need to check whether the condition

$$\Upsilon - A_\mu^{1/2} \geq 0$$

holds for all $\mu \in \{\tilde{\mu}\}$, with

$$\Upsilon = A_\nu^{1/2}|-\rangle\langle-| + A_{\pi-\nu}^{1/2}|+\rangle\langle+|. \quad (59)$$

Let us evaluate the entries of the operators $\Upsilon - A_\mu^{1/2}$ in the basis $|+\rangle, |-\rangle$. These are given by

$$\begin{aligned} \langle-|\Upsilon - A_\mu^{1/2}|-\rangle &= \langle-|A_\nu^{1/2}|-\rangle - \langle-|A_\mu^{1/2}|-\rangle \\ \langle+|\Upsilon - A_\mu^{1/2}|+\rangle &= \langle-|A_\nu^{1/2}|-\rangle - \langle+|A_\mu^{1/2}|+\rangle \\ \langle+|\Upsilon - A_\mu^{1/2}|-\rangle &= \langle+|A_\nu^{1/2}|-\rangle - \langle+|A_\mu^{1/2}|-\rangle, \end{aligned} \quad (60)$$

where we have used $\langle+|A_{\pi-\nu}^{1/2}|+\rangle = \langle-|A_\nu^{1/2}|-\rangle$. The eigenvalues $\lambda_1(\mu), \lambda_2(\mu)$ of the operator $\Upsilon - A_\mu^{1/2}$ can now be calculated from (60), and their positivity can be verified (at least numerically). The positivity of these eigenvalues (of which we do not report here the explicit expression) implies then (57). For all input states $|\psi_1\rangle$ discussed in this paper we have verified that the POVM given in (58), with ν given by (53), is indeed optimal.

From the above optimization procedure we see that the maximal fidelity, as a function of the parameter a , is given by

$$F_{\max}[m_1 = 0] = \frac{a \sqrt{1 - a^2} \cos \nu}{3\sqrt{3}} + \text{Tr } A_\nu^{1/2} + \text{Tr } A_{\pi/2}^{3/2}, \quad (61)$$

with ν given by (53). The fidelity as a function of the state parameter a is plotted in Fig. 1.

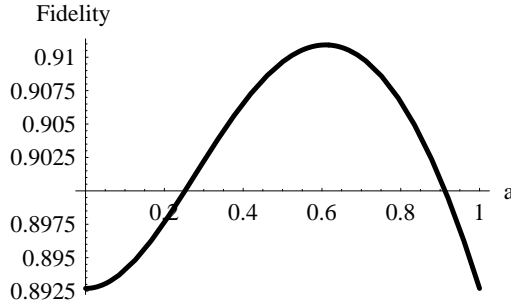


FIG. 1: Fidelity as a function of the state parameter a .

The maximal fidelity is achieved in correspondence of the state $|\psi_{\text{opt}}\rangle$, by setting $a = 0.609$, and is $F_{\text{max}}[\psi_{\text{opt}}] = 0.91092$. For comparison, the anti-parallel spin state

$$|\Phi_{\text{anti}}\rangle = |\uparrow\downarrow\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle$$

leads to a fidelity of $F_{\text{max}}[|\uparrow\downarrow\rangle] = 0.90982$, lower than the one obtained using the parallel spin state by only a factor $\sim 10^{-5}$. Similar results for parallel and anti-parallel spin states were obtained by N. Gisin and S. Iblisdir [24].

To conclude this part, we compare the above results to earlier results on quantum direction indicators, where a quantum system carrying a representation of the rotation group is used to transmit a spatial direction between two observers that do not share a common reference frame. As shown in [2, 6], if the state of the quantum system is constructed from two spin $1/2$, (*i.e.*, constrained to have maximal spin 1), encoding the directional information into anti-parallel spins proves to be the optimal strategy. Here we see instead that if the receiver is interested only in the relative orientation of this state with respect to another state, the anti-parallel spin state gives nearly the same fidelity as the parallel one, which is well below the optimum.

B. Higher values of j_2

The estimation of the relative orientation of two states can be seen as a process in which the first party (Alice) encodes a direction into a quantum state while the receiving party (Bob) attempts to estimate the signal without having a classical reference frame relative to which he can measure it. Therefore Bob resorts to finding the relative orientation of the signal state with respect to the orientation of some given state (say, a coherent state of spin

j_2), which serves as a quantum reference frame. So far the value of j_2 has been kept equal to $1/2$. We move on to consider what happens as we increase the value of j_2 . The limit $j_2 \rightarrow \infty$ could be regarded as the limit in which the quantum reference direction becomes a classical one.

As before, we need to consider states with $m_1 = 0$ and $m_1 = 1$. Different blocks of $\bar{\rho}(\beta)$ are found by coupling the representations of $|\psi_1\rangle$ and $|\psi_2\rangle$. For the $m_1 = 1$ sector, i.e. the case of parallel spins, $\bar{\rho}(\beta)$ has three one-dimensional blocks, since

$$\mathbf{1} \otimes \mathbf{j}_2 = (\mathbf{j}_2 - \mathbf{1}) \oplus \mathbf{j}_2 \oplus (\mathbf{j}_2 + \mathbf{1}).$$

The optimal measurement is then given by projections on subspaces of total angular momentum J .

For the $m_1 = 0$ sector we have

$$(\mathbf{0} \oplus \mathbf{1}) \otimes \mathbf{j}_2 = \mathbf{j}_2 \oplus (\mathbf{j}_2 - \mathbf{1}) \oplus \mathbf{j}_2 \oplus (\mathbf{j}_2 + \mathbf{1}),$$

thus $\bar{\rho}(\beta)$ and the operators A_μ have two 1-dimensional blocks and one 2-dimensional block. The POVM elements acting on the $J = j_2 - 1$ and $J = j_2 + 1$ subspaces are rank one projectors onto these subspaces. In the $J = j_2$ subspace, an optimization procedure similar to the one in Sec. III A needs to be done.

Carrying out the optimization, we find that the optimal state $|\psi_{\text{opt}}\rangle$ belongs to the $m_1 = 0$ sector for all values of j_2 . The optimal state is not fixed but rather depends on the value of j_2 . The limit $j_2 \rightarrow \infty$ yields the optimal asymptotic state

$$\lim_{j_2 \rightarrow \infty} |\psi_{\text{opt}}\rangle = a_\infty |00\rangle + \sqrt{1 - a_\infty^2} |10\rangle, \quad (62)$$

with $a_\infty = 0.595$. The dependence of the state $|\psi_{\text{opt}}\rangle$ on j_2 is plotted in Fig 2.

Comparing the fidelity achieved with the optimal state, as a function of j_2 , to the fidelity obtained using both the anti-parallel and the parallel spin states, we get, quite remarkably, almost the same fidelity for any value of j_2 . The parallel spins give a slightly higher fidelity, with the difference (already very small for $j_2 = 1/2$) rapidly decreasing for increasing values of j_2 . This comparison is plotted in Fig 3, which shows that the plot for the parallel and anti parallel spin states coincide.

Notice that although we expect the limit $j_2 \rightarrow \infty$ to be equivalent to measuring the state $|\psi_1\rangle$ against a classical reference direction, the anti-parallel spin states do not become

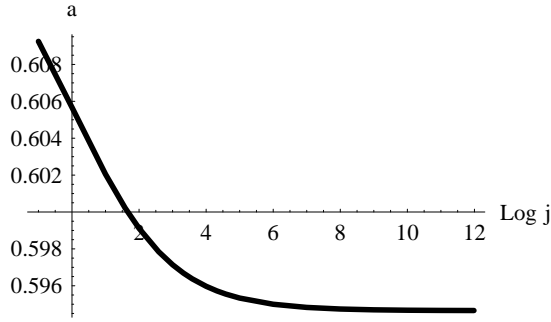


FIG. 2: The optimal state parameter a (vertical axis) with increasing values of j_2 (horizontal axis).

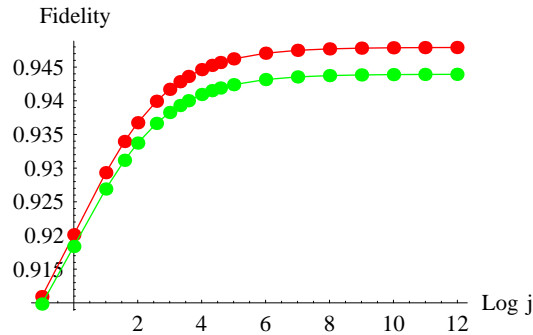


FIG. 3: Fidelity achieved using the optimal state (red, upper line), anti-parallel spins and parallel spins (green, lower line) as a function of j_2 .

optimal in this limit. This seems to be in contradiction with the result of [2, 6], who showed that if only two spins are available, the optimal direction indicator is provided by anti-parallel spins along that direction. The resolution to this apparent contradiction relies on the fact that the $j_2 \rightarrow \infty$ limit considered here is not equivalent to an estimation of a direction with respect to a classical reference frame, as [1, 2, 3, 4, 5, 6, 7, 8], but rather to that of an angle between the same vector, and, say, the z -axis of such a frame. In the next section we will consider the latter estimation task and show that it coincides with the limit $j_2 \rightarrow \infty$ discussed here, demonstrating that a macroscopic spin can be treated as a classical reference direction.

It is conceivable to consider a different estimation task in which one observer, say Alice, indicates a direction in space using the state $|\psi_1\rangle$, while Bob encodes his reference frame into the state $|\psi_2\rangle$, and finally a third observer is interested in the *orientation* of Alice's direction in Bob's frame. This task, however, cannot be performed using a spin coherent state $|\psi_2\rangle = |j_2, j_2\rangle$. It would be interesting to see what state would encode Bob's frame

in an optimal manner, and whether taking the appropriate limit would reproduce previous results for direction alignment.

C. Quantum-classical correspondence

Let us consider a scenario in which Alice prepares a state, $|\psi_1\rangle$, in order to indicate a chosen direction, and Bob is requested to estimate the angle between this direction and the z -axis of his *classical reference frame*, which replaces the quantum reference direction ($|\psi_2\rangle$) in the previous study. We assume no knowledge of Alice's reference frame, and without loss of generality we can assume that Alice chooses to indicate her z -axis. If the transformation relating Alice's frame to Bob's is parameterized by the Euler angles χ, β and ϕ , then, in the latter reference frame, Alice's state is given by $U(\chi, \beta, \phi)|\psi_1\rangle$. Bob's task is now to estimate the angle β between Alice's z -axis and his z -axis. As before, given $|\psi_1\rangle$, we are seeking a POVM that maximizes the fidelity

$$F\{E_\mu\} = \sum_\mu \int d_{\chi\beta\phi} \text{Tr}[\sigma_1(\chi, \beta, \phi) E_\mu] f(\mu, \beta), \quad (63)$$

where $d_{\chi\beta\phi} = \sin\beta d\beta d\phi d\chi/8\pi^2$ is the invariant measure of the rotation group and

$$\sigma_1(\chi, \beta, \phi) = U(\chi, \beta, \phi)|\psi_1\rangle\langle\psi_1|U(\chi, \beta, \phi)^\dagger.$$

Since we are not interested in estimating the angles ϕ and χ , we can integrate over them and define an averaged density matrix function of β only,

$$\bar{\sigma}(\beta) = \frac{1}{4\pi^2} \int d\phi d\chi \sigma_1(\chi, \beta, \phi). \quad (64)$$

Let us analyze the form of the density matrix $\bar{\sigma}(\beta)$. Writing $|\psi_1\rangle$ as in Eq. (14), and using the definition of the rotation operator matrix elements

$$\langle jm'|U(\chi, \beta, \phi)|jm\rangle = e^{im'\chi} d_{m'm}^j(\beta) e^{im\phi}, \quad (65)$$

we derive the matrix elements of $\bar{\sigma}(\beta)$ as

$$\langle j' m'|\bar{\sigma}(\beta)|j m\rangle = \sum_r \delta_{m'm} a_r^{j'} a_r^{j*} d_{m'r}^{j'}(\beta) d_{m'r}^j(\beta). \quad (66)$$

The matrix $\bar{\sigma}(\beta)$ is thus diagonal in the indices m , but has off-diagonal elements with different values of j . In this respect it is similar to the matrix $\bar{\rho}(\beta)$ defined in Eq. (30),

which was diagonal in the representation J with off-diagonal elements between different values of j_1 . Indeed, by taking the limit $j_2 \rightarrow \infty$ in Eq. (30) and interchanging the indices $J \rightarrow m$, we get the asymptotic equivalence between the matrices $\bar{\rho}(\beta)$ and $\bar{\sigma}(\beta)$, *i.e.*,

$$\lim_{j_2 \rightarrow \infty} \langle Jj_1 | \bar{\rho}(\beta) | Jj'_1 \rangle = \langle j_1 m | \bar{\sigma}(\beta) | j'_1 m \rangle. \quad (67)$$

This result shows that the fidelity achieved by relating any state $|\psi_1\rangle$ to a classical reference direction is identical to the one achieved in the limit $j_2 \rightarrow \infty$ discussed in the previous section (with the same state $|\psi_1\rangle$). Consequently, also the optimal state will be identical in the two cases.

IV. CONCLUDING REMARKS

In this work we studied the problem of estimating relative rotation angles of quantum signals. By considerations of global symmetry invariance, we derived the general form of the signal state and of the appropriate set of measurements and estimation strategies. For special low-dimensional cases, explicit optimization is carried out.

With these tools we have studied the state preparations which maximize the average fidelity of the estimation procedure, and compared these with the results found in the estimation of absolute rotations. We have also discussed the asymptotic limit in which one of the quantum states becomes a macroscopic spin. In this limit, the resulting estimation task is identical to an estimation of the orientation of a quantum state with respect to a classical reference direction.

Many important questions remain open for investigation. A broader extension of the problem would lead to the estimation of the orientation of a quantum state relative to another one that encodes a full reference frame (three axes), rather than a single direction. In such a scenario one has to estimate two angles (polar and azimuthal). An even more elaborate framework is one in which each quantum state encodes a full reference frame, and one is interested in estimating the transformation that would align these two reference frames. It would be interesting to compare the optimal states found in all these cases with those found in prior studies [3, 4, 5, 6, 7], when one of the reference frames is classical. We would like, finally, to emphasize that we did not include in our communication scheme the possibility of the two parties sharing a prior entangled state. In this case, the encoding and

the detection of relative information could proceed via a covariant dense coding scheme, which could increase the efficiency of the estimation.

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